Supplemental Material for
Galactic Axion Laser Interferometer Leveraging Electro-Optics: GALILEO

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Supporting information regarding thermal noise sources is provided in this Supplemental Material.

THERMAL NOISE SOURCES

The fluctuation-dissipation theorem (FDT) allows thermal noise contributions to a GALILEO measurement to be characterized according to the associated loss mechanism. Before estimating various noise contributions, let us briefly review the FDT. The temperature-dependent fluctuations of the effective interferometer arm length difference (i.e., effective displacement) z as our observable can be written as:

$$\delta z = \mathcal{A} \int \mathrm{d}^3\mathbf{r}\ \delta T(\mathbf{r}) q(\mathbf{r}) \tag{1}$$

where $q(\mathbf{r})$ is a space-dependent form factor and $\mathcal{A}$ is a coefficient that relates temperature to the observable of interest z. In this work, the form factor is proportional to the laser beam intensity profile in the transverse plane and is assumed constant along the beam line, i.e.,

$$q(\mathbf{r}) \propto I(x,y) = \frac{1}{\pi R_{\mathrm{beam}}^2} e^{-(x^2+y^2)/R_{\mathrm{beam}}^2}\,. \tag{2}$$

with R_{beam} being the Gaussian width of the optical beam intensity profile.

We are interested in calculating the noise power spectral density (PSD) $S_{\delta z} = \langle \delta z(\omega) \delta z^*(\omega) \rangle$ where ω is an arbitrary frequency within the range of light-DM masses relevant for the detector. According to FDT, one performs the following steps for each relevant loss mechanism:

- Assume a periodic, local entropy injection $\delta s_{\mathrm{inj}}(r)$ to the measurement system, with a density of

$$\frac{\delta s_{\mathrm{inj}}(\mathbf{r})}{\delta V} = \frac{1}{T}\frac{\delta Q_{\mathrm{inj}}(\mathbf{r})}{\delta V} = F_0 \cos(\omega t) q(\mathbf{r}) \tag{3}$$

  where δQ_{inj} is the injected heat/energy, and T is the mean temperature. If z has dimensions of length, then F_0 has dimensions of force, with a magnitude set by the specific loss mechanism being considered.

- Calculate the dissipated power W_{diss} associated with each loss mechanism considered.

- Compute the the noise PSD for each loss mechanism via

$$S_{\delta z}(\omega) = \mathcal{A}^2 \frac{8 k_B T}{\omega^2} \frac{\langle W_{\mathrm{diss}} \rangle}{F_0^2} \tag{4}$$

  where k_B is the Boltzmann constant and $\langle \cdots \rangle$ denotes time-averaging over one cycle with period $2\pi/\omega$.

Table I provides benchmark experimental parameters. Note that we choose the hierarchy $L_z < R_{\mathrm{beam}} \ll L_{x,y}$, which is analogous to the geometry of LIGO mirror coatings; the calculations that follow will also be analogous.

We will compare different thermal noise contributions with quantum (photon shot) noise, assuming lithium niobate ($LiNbO_3$) as the electro-optic (EO) material, with the amplitude spectral density (ASD) of the quantum noise (see the main text for details):

$$\mathrm{ASD}_{\mathrm{s.n.}} = \sqrt{\hbar \omega_{\mathrm{laser}} P_{\mathrm{in}}} = 9.7 \times 10^{-10}\, \frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}\,. \tag{5}$$

Using the transfer function $\delta P_{\mathrm{out}}/\delta L = (2\pi/\lambda) P_{\mathrm{in}} N \mathcal{F} \sim 2 \times 10^{13}\,\mathrm{W/m}$ we can translate this ASD to an effective displacement ASD:

$$\mathrm{ASD}_{\mathrm{s.n.}} = \sqrt{\mathcal{S}_{\delta z}} \sim 4 \times 10^{-23}\, \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \tag{6}$$

According to the following computations of the thermal noise sources for an operational instrument temperature $< 200\,\mathrm{K}$, we conclude that in the DM mass range of interest:

$$\mathrm{ASD}_{\mathrm{thermal}} < \mathrm{ASD}_{\mathrm{s.n.}} \tag{7}$$

Parameter	Value	Description
$\mathcal{F}$	$\sim 10^5$	Cavity finesse
P	5 W	Laser power
λ	1064 nm	Laser wavelength
$R_{\rm beam}$	1 cm	Gaussian width of the laser beam
T	200 K	Cavity temperature
L_z	$L_0 \leq 6.7$ mm	EO material thickness
$L_{x,y}$	5 cm	EO material lateral dimensions
σ	0.25	Poisson ratio*
Y_0	170 GPa	Young's modulus*
φ	10^{-6}	Loss angle*
ρ	$4.64\,\mathrm{g/cm^3}$	Density*
κ	$4.6\,\mathrm{W/(m\cdot K)}$	Thermal conductivity*
C	$628\,\mathrm{J/(kg\cdot K)}$	Heat capacity*
α	$14.8\times10^{-6}/\mathrm{K}$	Thermal expansion*
β	$\sim 5\times10^{-4}/\mathrm{K}$	Thermo-refractive coefficient*

*The material properties are reported for $LiNbO_3$.

TABLE I. Benchmark experimental parameters.

Brownian noise

Homogeneous mechanical damping inside a material is typically characterized by an imaginary contribution to the Young's modulus:

$$Y = Y_0[1 + i\varphi(\omega)] \tag{8}$$

where φ is the so-called loss angle, which is given by the quality factor of the material's internal mechanical modes as $\varphi = 1/Q_{\rm int}$. The dissipated power due to φ can be written as $\left\langle W_{\rm diss|mechanical}\right\rangle = \omega\varphi(\omega)U_{\rm def}$, where $U_{\rm def}$ is the maximum elastic deformation energy of the material under an oscillatory drive at frequency ω. Assuming that $L_{x,y} \gg R_{\rm beam}$ and that the mechanical resonance frequencies are far away from ω, one recovers the well-known result for the Brownian noise PSD as studied in the context of interferometric gravitational wave detectors [96]:

$$S_{\delta z|\rm Brownian} \simeq 1.87 \times \frac{8k_BT}{\omega}\frac{1-\sigma^2}{\pi^2 Y_0 R_{\rm beam}}\varphi \tag{9}$$

where the numerical factor 1.87 applies to a Gaussian optical beam. Therefore, for $LiNbO_3$ EO material in the GALILEO detector we have:

$$\mathrm{ASD}_{\rm Brownian} = \sqrt{\mathrm{S}_{\delta z|\rm Brownian}} \simeq 6.1\times10^{-24}\left(\frac{2\pi\times 10\,\mathrm{GHz}}{\omega}\right)^{1/2}\frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \tag{10}$$

for the choice of parameters in Table I.

Inhomogeneous thermal noise

Whenever there is space-dependent temperature variations $\delta T(\mathbf{r})$, there will be entropy redistribution, and therefore dissipation, due to heat flow. The rate of change of entropy density s in terms of heat flow per unit area $\mathbf{j}$ is given by $\dot{s} = -(\nabla\cdot\mathbf{j})/T$, which implies:

$$W_{\rm diss} = -T\int \mathrm{d}^3\mathbf{r}\frac{\nabla\cdot\mathbf{j}}{T} = \underbrace{-T\int \mathrm{d}^2\vec{A}\cdot\frac{\mathbf{j}}{T}}_{\rm surface}\underbrace{-T\int \mathrm{d}^3\mathbf{r}\;\mathbf{j}\cdot\nabla(\frac{1}{T})}_{\rm bulk} \tag{11}$$

where we used $W_{\rm diss} = T\dot{S}$. Using κ as the thermal conductivity in the bulk, i.e., $\mathbf{j} = -\kappa\nabla T$, the second term can be written as:

$$W_{\rm diss|bulk} = \frac{\kappa}{T}\int \mathrm{d}^3\mathbf{r}\;(\nabla\delta T)^2 \tag{12}$$

To proceed with the surface term, we assume radiative thermal equilibrium with the surrounding environment, which implies a vanishing net radiative heat flow. However, at first order we have $\mathbf{j} = 4\epsilon_{\rm emm}\sigma_{\rm SB}T^3\delta T$ (linearized Stefan-Boltzmann law), where $\epsilon_{\rm emm}$ and $\sigma_{\rm SB}$ are the EO material's emissivity and Stefan-Boltzmann constant, respectively. As a result, the radiative surface dissipation is:

$$W_{\rm diss|surface} = \frac{1}{T}\int \mathrm{d}^2\vec{A}\cdot\mathbf{j}\delta T = 4\epsilon_{\rm emm}\sigma_{\rm SB}T^2\int \mathrm{d}x\mathrm{d}y\ \delta T^2 \tag{13}$$

Next, we employ the process of entropy injection according to Eq. (3), so that we can use the FDT to compute the noise PSD. We solve the diffusion equation with this source:

$$\rho C\dot{\delta T} - \kappa\nabla^2\delta T = \partial_t\left(\frac{\partial Q_{\rm inj}}{\partial V}\right) \tag{14}$$

which can be greatly simplified considering the hierarchy of length scales involved in the problem. We notice that the sharpest spatial heat injection gradient in the system is given by the laser beam radius. For the benchmark parameters given in Table I, we can thus neglect the second term in the diffusion equation. As a result,

$$\delta T = \frac{1}{\rho C}TF_0\cos(\omega t)q(\mathbf{r}) \tag{15}$$

with $q(\mathbf{r}) = (1/L_z)I(x,y)$. The bulk and surface dissipated power, following the above integrals, are [100]:

$$\left\langle W_{\rm diss|bulk}\right\rangle = \frac{\kappa T F_0^2}{2\rho^2C^2}\frac{1}{L_z}\frac{1}{\pi R_{\rm beam}^4} \tag{16}$$

$$\left\langle W_{\rm diss|surface}\right\rangle = \frac{2\epsilon_{\rm emm}\sigma_{\rm SB}T^4F_0^2}{\rho^2C^2}\frac{1}{L_z^2}\frac{1}{\pi R_{\rm beam}^2} \tag{17}$$

Thermo-elastic effect. Temperature fluctuations can couple to the interferometer phase noise through thermal expansion [97]. For a non-zero thermal expansion coefficient $\alpha \neq 0$,

$$\delta z \equiv \delta L_z = \alpha(1+\sigma)L_z\delta T \tag{18}$$

where, σ is the Poisson ratio. Therefore, the $\mathcal{A}$ coefficient in the FDT for thermo-elastic noise source is:

$$\mathcal{A}_{\rm TE} = \alpha(1+\sigma)L_z \tag{19}$$

Thermo-refractive effect. Temperature fluctuations can also couple to the interferometer phase noise through the temperature dependence of the refractive index [98]. In other words, if $\beta = \mathrm{d}n/\mathrm{d}T \neq 0$,

$$\delta z \equiv \delta L_z = -\beta\lambda\delta T \tag{20}$$

In this case, the corresponding $\mathcal{A}$ coefficient is:

$$\mathcal{A}_{\rm TR} = -\beta\lambda \tag{21}$$

As pointed out in [99], we should consider thermo-elastic and thermo-refractive effects coherently. We thus define a total coefficient:

$$\mathcal{A} = \mathcal{A}_{TE} + \mathcal{A}_{TR} = \alpha(1+\sigma)L_z - \beta\lambda \tag{22}$$

Finally, we are ready to combine all results into the FDT, and compute the bulk and surface contributions to the interferometer noise:

$$S_{\delta z|\rm bulk} = \mathcal{A}^2\frac{4k_B\kappa}{\omega^2}\frac{T^2}{\rho^2C^2}\frac{1}{L_z}\frac{1}{\pi R_{\rm beam}^4} \tag{23}$$

$$S_{\delta z|\rm surface} = \mathcal{A}^2\frac{16k_B\epsilon_{\rm emm}\sigma_{\rm SB}}{\omega^2}\frac{T^5}{\rho^2C^2}\frac{1}{L_z^2}\frac{1}{\pi R_{\rm beam}^2} \tag{24}$$

which implies,

$$\mathrm{ASD}_{\rm bulk} = \sqrt{\mathrm{S}_{\delta z|\rm bulk}} = 5.6\times10^{-29}\left(\frac{2\pi\times10\,\mathrm{GHz}}{\omega}\right)\frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \tag{25}$$

$$\mathrm{ASD}_{\rm surface} = \sqrt{\mathrm{S}_{\delta z|\rm surface}} = 1.1\times10^{-29}\left(\frac{2\pi\times10\,\mathrm{GHz}}{\omega}\right)\frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \tag{26}$$

for the choice of parameters given in Table I and $\epsilon_{\rm emm} = 1$.

Photo-thermal noise

Another source of temperature fluctuation inside the EO material is photon absorption. Optical photons absorbed in the material can generate a collection of phonons that thermalize quickly and lead to local temperature increase [97]. The noise PSD due to such a process in a thin material is calculated by Braginsky et al.:

$$S_{\delta z|\mathrm{photo-thermal}} = \alpha^2(1+\sigma)^2 \frac{\hbar\omega_{\mathrm{laser}} W_{\mathrm{abs}}}{(\rho C \pi R_{\mathrm{beam}}^2)^2} \frac{1}{\omega^2} \tag{27}$$

where ω_{laser} and W_{abs} are the carrier frequency and average absorbed power. We assume that 10% of the total power gets absorbed in the medium. In this case, we have:

$$\mathrm{ASD}_{\mathrm{photo-thermal}} = \sqrt{\mathrm{S}_{\delta z|\mathrm{photo-thermal}}} \simeq 1.3 \times 10^{-28} \left(\frac{2\pi \times 10\,\mathrm{GHz}}{\omega}\right) \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \tag{28}$$

for the choice of parameters given in Table I.